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Practical Exponential Coordinates using Implicit Dual Quaternions

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Abstract. Modern approaches for robot kinematics employ the product of exponentials formulation, represented using homogeneous transformation matrices. Quaternions over dual numbers are an established alternative representation; however, their use presents certain challenges: the dual quaternion exponential and logarithm contain a zero-angle singularity, and many common operations are less efficient using dual quaternions than with matrices. We present a new derivation of the dual quaternion exponential and logarithm that removes the singularity, and we show an implicit representation of dual quaternions offers analytical and empirical efficiency advantages compared to both matrices and explicit dual quaternions. Analytically, implicit dual quaternions are more compact and require fewer arithmetic instructions for common operations, including chaining and exponentials. Empirically, we demonstrate a 25%-40% speedup to compute the forward kinematics of multiple robots. This work offers a practical connection between dual quaternions and modern exponential coordinates, demonstrating that a quaternion-based approach provides a more efficient alternative to matrices for robot kinematics.

1 Introduction

Efficient geometric computations are important for robot manipulation, 3D simulation, and other areas that must represent the physical world. The product of exponentials formulation, represented as homogeneous transformation matrices, has emerged as the conventional method for robot kinematics [2,14,16]. For pure rotations, the unit quaternion has recently resurged in popularity, particularly for applications in graphics and estimation where the efficient interpolation and normalization of quaternions is especially useful. It is also possible to represent both rotation and translation by extending the ordinary unit quaternion to quaternions over dual numbers [18,22]. Such dual quaternions retain the unit quaternions' advantages of compactness and efficient normalization; however, they also present challenges. Common kinematics operations—constructing and chaining transforms—require more arithmetic instructions using dual quaternions than the equivalent transformation matrix computation. Critically, the dual quaternion exponential contains a small-angle singularity which must be handled for numerical robustness. We address these challenges and present a dual-quaternion-based representation with advantages for robot kinematics.

We present a new derivation of the dual number quaternion exponential and logarithm that removes the small-angle singularity, and we show that the implicit representation of a dual quaternion is more computationally-efficient for robot kinematics than homogeneous transformation matrices. The conventional representation of exponential coordinates using the homogeneous transformation matrix provides a baseline for comparison (see Sec. 3). We begin with the known forms of the ordinary quaternion exponential and logarithm (see Sec. 4.1). Then, based on dual number arithmetic and quaternion geometry, we derive the exponential and logarithm for the dual quaternions and rewrite factors to identify Taylor series that remove the singularities (see Sec. 4.2). We extend this dual quaternion exponential and logarithm to the implicit representation of a dual quaternion as an ordinary (rotation) quaternion and a translation vector, which is more compact and computationally efficient than explicit dual quaternions (see Sec. 4.3). Next, we present the application of these quaternion forms to robot kinematics, demonstrating a 25%-40% empirical performance gain over transformation matrices. Finally, we discuss issues of equivalence and efficiency between matrix and quaternion representations (see Sec. 6).

All the derived forms presented in this paper are available as open source software.[1]

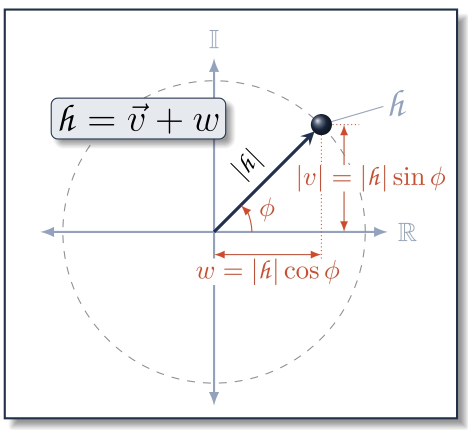


Fig. 1. The quaternion-imaginary-plane, containing axes for the scalar w and vector magnitude $|v|$.

Quaternion-based forms present both challenges and advantages. A common challenge raised with quaternions is the difficulty of mentally visualizing the four-dimensional space of ordinary quaternions—or the eight-dimensional space of dual quaternions—whereas vector and matrix approaches have a direct, 3-dimensional interpretation. Still, the planar projection of quaternions (see Fig. 1) provides some insight into the relationship between quaternion components and angles. More importantly, a growing body of work continues to demonstrate that ordinary and dual quaternions offer computational advantages in a variety of domains [11,15,20]. The results of this paper are in the same vein. We demononstrate a dual-quaternion-based approach for the product of exponentials that offers computational advantages over matrices. We mitigate the challenge of visualizing quaternions by using the relations of Fig. 1 in an algebraic derivation.

A key technique in our derivation is to rewrite factors with singularities into forms with well-defined Taylor series which we can use for evaluation near the singularity. Grassia applies this idea to ordinary quaternions [8]. For example, the

[1] Software available at `http://amino.dyalab.org`

ordinary quaternion exponential contains the factor $\frac{\sin\theta}{\theta}$, which has a singularity at $\theta = 0$. However, we can use a Taylor series to remove the singularity:

$$\frac{\sin\theta}{\theta} = 1 - \frac{1}{6}\theta^2 + \frac{1}{120}\theta^4 - \frac{1}{5040}\theta^6 + \ldots \qquad \Longrightarrow \qquad \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\,. \tag{1}$$

Near the singularity, we need only the initial terms of the Taylor series to evaluate the factor to within machine precision because the final terms will be smaller than the precision we can represent. For (1), we have alternating positive and negative terms of decreasing magnitude, so the error after evaluating the first i terms is no greater than the magnitude of term $i+1$. We need not evaluate any additional terms with when this error is less than machine precision. For example, when θ^4 is less than machine precision, we may can achieve minimum possible numerical error using only the first two terms $1 - \frac{1}{6}\theta^2$.

We extend this Taylor series construction to the dual quaternions, which have similar—though more complicated—factors containing singularities. We use quaternion trigonometry (see Fig. 1) to rewrite these factors into forms that are defined in the limit via Taylor series. Table 1 lists several common Taylor series.

$\sin\theta$	$= \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \ldots$
$\cos\theta$	$= 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \ldots$
$\frac{\sin\theta}{\theta}$	$= 1 - \frac{1}{6}\theta^2 + \frac{1}{120}\theta^4 + \ldots$
$\frac{\theta}{\sin\theta}$	$= 1 + \frac{1}{6}\theta^2 + \frac{7}{360}\theta^4 + \ldots$
$\frac{1-\cos\theta}{\theta^2}$	$= \frac{1}{2} - \frac{1}{24}\theta^2 + \frac{1}{720}\theta^4 + \ldots$

Table 1. Taylor Series for $\theta \to 0$

We use the following notation. Bold uppercase $\mathbf{R}$ denotes a matrix. Bold lowercase $\mathbf{v}$ denotes a vector. An over-arrow $\vec{v}$ denotes a length-three vector over the basis units $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$. An over-hat $\hat{\boldsymbol{u}}$ denotes a unit vector ($|u| = 1$). An over-tilde $\tilde{n}$ denotes a dual number ($\tilde{n} = n_{\text{real}} + n_{\text{dual}}\varepsilon$). The lowercase script $\mathscr{h}$ denotes an ordinary quaternion. The uppercase script $\mathcal{S}$ denotes a dual quaternion ($\mathcal{S} = \mathscr{s}_{\text{real}} + \mathscr{s}_{\text{dual}}\varepsilon$). We abbreviate sin and cos with $\mathscr{s}$ and $\mathscr{c}$.

2 Related Work

Brockett connected robot kinematics with Lie groups expressed as matrix exponentials [2]. This product of exponentials formulation has become the conventional approach for robot kinematics [14,16]. Our work presents a practical connection between such exponential coordinates and quaternion-based representations, and we show that a quaternion-based representation can offer efficiency advantages compared to matrix-based representations.

Quaternions provide an alternative to matrix-based geometric representations. Unit quaternions represent rotations [9] with four elements: a 3-element vector and a scalar that together encode the rotational axis, and the sine and cosine of the rotational angle. Though vector analysis became the preferred notation in many areas [1,7], quaternions have seen renewed use in recent years as a practical representation for rotation, interpolation, and estimation [8,13,15,20]. The computational advantages of quaternions in such applications suggest that a quaternion-based approach could merit investigation in other areas typically addressed using vector or matrix representations.

Quaternions over dual numbers—the *dual quaternion*—can represent both rotation and translation [22,23]. Selig presents a modern context for dual quaternions and more broadly Clifford algebras in relation to Lie algebras [18]. Yang and Freudenstein applied dual quaternions to the analysis of spatial mechanisms (closed chains) [27]. Several recent authors have applied dual quaternions to robot kinematics [4,5,10,12,17,21,25]. Wang, et. al. compare dual quaternion and homogeneous matrix approaches, showing that dual quaternions are often more efficient [26]. We continue this application of dual quaternions to robot kinematics by addressing issues of singularities and numerical robustness in the dual quaternion exponential and logarithm.

Though the form of the dual quaternion exponential is well established [6,19], there is, to our knowledge, no prior work that addresses the zero-angle singularity in the dual quaternion exponential and logarithm, which is necessary to practically use dual quaternions in the product of exponentials formulation. Han, et. al. observe, though do not address, the zero-angle discontinuity [10]. Wang, et. al. provide an approximation of the logarithm [25]. In this work, we present new, exact derivations of the dual quaternion exponential and logarithm that remove the zero-angle singularity, enabling the practical use of dual quaternions as exponential coordinates. Furthermore, we show that implicitly representing a dual quaternion as an ordinary quaternion and a translation vector is both more compact and more computationally efficient for common kinematics operations than either explicit dual quaternions or homogeneous transformation matrices.

3 Rotation and Transformation Matrix Maps

We briefly restate the key matrix operations for robot kinematics to compare against the quaternion forms and illustrate the Taylor series construction.

3.1 Rotation Matrix

We define the rotation matrix exponential and logarithm using the *rotation vector*, i.e., the rotation axis scaled by the rotation angle, because separating the axis and angle results in an undefined axis when the angle is zero and poor numerical stability when attempting to construct the unit axis for small angles [8].

The rotation matrix exponential [14] is:

$$\mathrm{e}^{[\omega]} = \mathbf{I} + \frac{\sin|\omega|}{|\omega|}[\omega] + \frac{1-\cos|\omega|}{|\omega|^2}[\omega]^2, \quad \text{where} \quad [\omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} . \quad (2)$$

We remove the singularity at $|\omega| = 0$ via the Taylor series in Table 1 for $\frac{\sin|\omega|}{|\omega|}$ and $\frac{1-\cos|\omega|}{|\omega|^2}$.

The rotation matrix logarithm [14] is:

$$\vec{\omega} = \frac{\theta}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{30} \\ r_{21} - r_{12} \end{bmatrix}, \qquad \text{where} \qquad \theta = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) . \quad (3)$$

We remove the singularity at $\theta = 0$ via the Taylor series in Table 1 for $\frac{\theta}{\sin\theta}$.

3.2 Transformation Matrix:

The transformation matrix exponential [14] is:

$$\exp\begin{pmatrix}\vec{\omega}\\ \vec{\nu}\end{pmatrix} = \begin{bmatrix} e^{[\omega]} & \left(\mathbf{I} + \frac{1-\cos|\omega|}{|\omega|}[\omega] + \frac{1-\frac{\sin|\omega|}{|\omega|}}{|\omega|^2}[\omega]^2\right)\vec{\nu} \\ \mathbf{0} & 1 \end{bmatrix} . \tag{4}$$

We remove the singularity at $|\omega| = 0$ via the Taylor series in Table 1 for $\frac{1-\cos|\omega|}{|\omega|^2}$ and the following:

$$\frac{1-\frac{\sin|\omega|}{|\omega|}}{|\omega|^2} = \frac{1}{6} - \frac{|\omega|^2}{120} + \frac{|\omega|^4}{5040} + \dots \tag{5}$$

The transformation matrix logarithm [14] is given by:

$$\ln\begin{bmatrix}\mathbf{R} & \mathbf{v}\\ \mathbf{0} & 1\end{bmatrix} = \begin{pmatrix} \ln\mathbf{R} \\ \left(\mathbf{I} - \frac{[\omega]}{2} + \frac{2s-|\omega|(1+c)}{2s|\omega|^2}[\omega]^2\right)\mathbf{v}\end{pmatrix} . \tag{6}$$

We remove the singularity at $|\omega| = 0$ via the following Taylor series:

$$\frac{2\sin|\omega| - |\omega|\left(1+\cos|\omega|\right)}{2\left(\sin|\omega|\right)|\omega|^2} = \frac{1}{12} + \frac{|\omega|^2}{720} + \frac{|\omega|^4}{30240} + \dots \tag{7}$$

4 Ordinary, Dual, and Implicit Quaternion Maps

Now, we present the key contribution of this work: new, singularity free forms of the dual quaternion exponential and logarithm and their corresponding forms for the implicit, quaternion-translation representation. Our derivation starts with the established ordinary quaterion exponential and logarithm (see Sec. 4.1). Then, we derive the dual quaternion forms (see Sec. 4.2) by using the quaternion trigonometry (see Fig. 1) to construct Taylor series that remove the singularities. Finally, we derive the equivalent exponential and logarithm for the more compact and efficient quaternion-translation representation (see Sec. 4.3).

4.1 Ordinary Quaternions

Ordinary quaternions extend complex numbers ($\hat{\boldsymbol{\imath}}^2 = -1$) to three units:

$$\hat{\boldsymbol{\imath}}^2 = \hat{\boldsymbol{\jmath}}^2 = \hat{\boldsymbol{k}}^2 = \hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{k}} = -1 . \tag{8}$$

A quaternion, therefore, has four elements: the real term (scalar) and the coefficients of each quaternion unit $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ (vector). We use the following notations for the quaternion elements:

$$\hbar = \underbrace{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}}_{\text{vector } \vec{v}} + \underbrace{w}_{\text{scalar}} = \vec{v} + w \; . \tag{9}$$

The dot ($\cdot$) and cross ($\times$) products, though actually introduced as an alternative to quaternions [7], allow a compact definition of the quaternion multiplication ($\otimes$):

$$q \otimes p = \underbrace{\vec{q}_v \times \vec{p}_v + q_w \vec{p}_v + p_w \vec{q}_v}_{\text{vector } \vec{v}} + \underbrace{q_w p_w - \vec{q}_v \cdot \vec{p}_v}_{\text{scalar}} \; . \tag{10}$$

The quaternion conjugate and rotation operation are:

$$\hbar^* = -\vec{\hbar}_v + \hbar_w \qquad \text{and} \qquad {}^a p = {}^a\hbar_b \otimes {}^b p \otimes {}^a\hbar_b^* \; . \tag{11}$$

The quaternion exponential is:

$$\mathrm{e}^{\vec{v}+w} = \mathrm{e}^{w} \left(\left(\frac{\sin|v|}{|v|} \right) \vec{v} + \cos|v| \right) \; . \tag{12}$$

When $|v|$ approaches zero, we can use the Taylor series for $\frac{\sin|v|}{|v|}$ in Table 1.

To compute the logarithm, we first find the angle between the vector $\vec{v}$ and scalar w parts of the quaternion. Then the logarithm is as follows:

$$\phi = \operatorname{atan2}(|v|, w) \quad \text{and} \quad \ln \hbar = \frac{\phi}{|v|}\vec{v} + \ln|\hbar| \; . \tag{13}$$

When $|v|$ approaches zero, we handle the singularity in $\frac{\phi}{|v|}$ by rewriting as follows:

$$\frac{\phi}{|v|} = \frac{\frac{\phi}{|\hbar|}}{\frac{|v|}{|\hbar|}} = \frac{\frac{\phi}{|\hbar|}}{\sin\phi} = \frac{\frac{\phi}{\sin\phi}}{|\hbar|} = \frac{1 + \frac{\theta^2}{6} + \frac{7\theta^4}{360} + \frac{31\theta^6}{15120} + \dots}{|\hbar|} \; . \tag{14}$$

4.2 Dual Quaternion

Dual quaternions are a compact representation that offers useful analytic properties. We briefly review the use of dual quaternions for kinematics before introducing our new derivations of the exponential and logarithm to handle the small-angle singularity. For a more thorough overview of dual quaternions for kinematics, please see [18].

Dual quaternions combine ordinary quaternions with the dual numbers, $\boldsymbol{\varepsilon}$, defined as:

$$\boldsymbol{\varepsilon}^2 = 0 \qquad \text{and} \qquad \boldsymbol{\varepsilon} \neq 0 \; . \tag{15}$$

A dual quaternion will thus have eight coefficients covering all combinations of the quaternion elements, dual element, and scalars. We write a dual quaternion as:

$$\mathcal{S} = \hbar + \text{đ}\boldsymbol{\varepsilon} = \hbar_x\hat{\boldsymbol{\imath}} + \hbar_y\hat{\boldsymbol{\jmath}} + \hbar_z\hat{\boldsymbol{k}} + \hbar_w + \left(\text{đ}_x\hat{\boldsymbol{\imath}} + \text{đ}_y\hat{\boldsymbol{\jmath}} + \text{đ}_z\hat{\boldsymbol{k}} + \text{đ}_w\right)\boldsymbol{\varepsilon}\ . \tag{16}$$

The Euclidean transformation consisting of rotation $\hbar$ and translation $\vec{v}$ corresponds to the following dual quaternion.

$$\mathcal{S} = \hbar + \text{đ}\boldsymbol{\varepsilon} = \hbar + \frac{1}{2}\vec{v} \otimes \hbar\boldsymbol{\varepsilon} \qquad \text{and} \qquad \vec{v} = 2\text{đ} \otimes \hbar^*\ . \tag{17}$$

Multiplication of dual quaternions will chain successive transforms.

$$\begin{aligned} {}^a\mathcal{S}_c &= {}^a\mathcal{S}_b \otimes {}^b\mathcal{S}_c = \left({}^a\hbar_b + {}^a\text{đ}_b\boldsymbol{\varepsilon}\right) \otimes \left({}^b\hbar_c + {}^b\text{đ}_c\boldsymbol{\varepsilon}\right) \\ &= {}^a\hbar_b \otimes {}^b\hbar_c + \left({}^a\hbar_b \otimes {}^b\text{đ}_c + {}^a\text{đ}_b \otimes {}^b\hbar_c\right)\boldsymbol{\varepsilon}\ . \end{aligned} \tag{18}$$

Rewriting (18) in terms of a transformation and a point yields the dual quaternion form to transform a point. An equivalent derivation extends (11) to the dual numbers.

$${}^a\mathcal{S}_c = {}^a\mathcal{S}_b \otimes \left(1 + {}^bp\boldsymbol{\varepsilon}\right) \qquad \rightsquigarrow \qquad {}^ap = \left(2\text{đ} + \hbar \otimes {}^bp\right) \otimes \hbar^* \tag{19}$$

The Taylor series for functions of dual numbers yields a useful property: all higher-order terms containing $\boldsymbol{\varepsilon}^2$ cancel to zero.

$$\begin{aligned} f(r + d\boldsymbol{\varepsilon}) &= f(r) + \frac{f'(r)}{1!}(d\boldsymbol{\varepsilon}) + \frac{f''(r)}{2!}\cancelto{0}{(d\boldsymbol{\varepsilon})^2} + \frac{f'''(r)}{3!}\cancelto{0}{(d\boldsymbol{\varepsilon})^3} + \cancelto{0}{\ldots} \\ &= f(r) + \boldsymbol{\varepsilon} d f'(r)\ . \end{aligned} \tag{20}$$

The dual number Taylor series (20) enables evaluation of dual number functions using only the value and derivative of the real function. We summarize several relevant dual functions in Table 2.

$f(r + d\epsilon) =$	$f(r) + \boldsymbol{\varepsilon} d(f'(r))$
$\cos(r + d\boldsymbol{\varepsilon}) =$	$\cos r - \boldsymbol{\varepsilon} d \sin r$
$\sin(r + d\boldsymbol{\varepsilon}) =$	$\sin r + \boldsymbol{\varepsilon} d \cos r$
$\tan^{-1}(r + d\boldsymbol{\varepsilon}) =$	$\tan^{-1} r + \frac{\boldsymbol{\varepsilon} d}{r^2+1}$
$\exp(r + d\boldsymbol{\varepsilon}) =$	$\mathrm{e}^r + \boldsymbol{\varepsilon}\mathrm{e}^r d$
$\ln(r + d\boldsymbol{\varepsilon}) =$	$\ln r + \frac{d}{r}\boldsymbol{\varepsilon}$
$\sqrt{r + d\boldsymbol{\varepsilon}} =$	$\sqrt{r} + \boldsymbol{\varepsilon}\frac{d}{2\sqrt{r}}$

Table 2. Dual numbers functions

Singularity-Free Dual Quaternion Exponential To derive a suitable form of the dual quaternion exponential, we begin by rewriting the ordinary quaternion exponential (12) over dual numbers.

$$\begin{aligned} \tilde{\phi} &= \sqrt{(\hbar_x + \text{đ}_x\boldsymbol{\varepsilon})^2 + (\hbar_y + \text{đ}_z\boldsymbol{\varepsilon})^2 + (\hbar_y + \text{đ}_z\boldsymbol{\varepsilon})^2} \\ e^{\mathcal{S}} &= \mathrm{e}^{\hbar_w + \text{đ}_w\boldsymbol{\varepsilon}}\left(\frac{\sin\tilde{\phi}}{\tilde{\phi}}\left(\left(\hbar_x + \text{đ}_x\boldsymbol{\varepsilon}\right)\hat{\boldsymbol{\imath}} + \left(\hbar_y + \text{đ}_y\boldsymbol{\varepsilon}\right)\hat{\boldsymbol{\jmath}} + \left(\hbar_y + \text{đ}_z\boldsymbol{\varepsilon}\right)\hat{\boldsymbol{k}}\right) + \cos\tilde{\phi}\right) \end{aligned} \tag{21}$$

Direct evaluation of (21) must contend with the singularity (zero denominator) in the factor $\frac{\sin\tilde{\phi}}{\tilde{\phi}}$. To handle the singularity, we will algebraically expand the dual arithmetic and rewrite factors based on quaternion trigonometry into forms where we can find suitable Taylor series.

First, we expand the dual quaternion angle $\tilde{\phi}$.

$$\begin{aligned}\tilde{\phi} &= \sqrt{(h_x + d_x\varepsilon)^2 + (h_y + d_z\varepsilon)^2 + (h_y + d_z\varepsilon)^2} \\ &= \sqrt{h_x^2 + h_y^2 + h_y^2} + \frac{h_x d_x + h_y d_y + h_y d_z}{\sqrt{h_x^2 + h_y^2 + h_y^2}}\varepsilon = \phi + \frac{\gamma}{\phi}\varepsilon \end{aligned} \tag{22}$$

where ϕ is the same as the ordinary quaternion angle and $\gamma = \vec{h}_v \cdot \vec{d}_v$.

The dual sin and cos are then

$$\cos\tilde{\phi} = c - \frac{\gamma}{\phi}s\varepsilon \qquad \text{and} \qquad \sin\tilde{\phi} = s + \frac{\gamma}{\phi}c\varepsilon\ . \tag{23}$$

where $s = \sin\phi$ and $c = \cos\phi$.

Next, we expand the dual sinc function $\frac{\sin\tilde{\phi}}{\tilde{\phi}}$ and rearrange terms to find a suitable Taylor series to handle the singularity at $\phi = 0$:

$$\begin{aligned}\frac{\sin\tilde{\phi}}{\tilde{\phi}} &= \frac{\sin(\phi) + \frac{\gamma}{\phi}\cos(\phi)\varepsilon}{\phi + \frac{\gamma}{\phi}\varepsilon} = \frac{\sin(\phi)}{\phi} + \gamma\left(\frac{\cos(\phi) - \frac{\sin(\phi)}{\phi}}{\phi^2}\right)\varepsilon \\ &= \underbrace{\left(1 - \frac{\phi^2}{6} + \frac{\phi^4}{120} + \dots\right)}_{(\sin\phi)/\phi} + \gamma\underbrace{\left(-\frac{1}{3} + \frac{\phi^2}{30} - \frac{\phi^4}{840} + \dots\right)}_{(\cos\phi - (\sin\phi)/\phi)/\phi^2}\varepsilon\ . \end{aligned} \tag{24}$$

Finally, we expand the original form of the exponential in (21):

$$\mathrm{e}^{\mathcal{S}} = \left(\mathrm{e}^{h_w} + d_w\mathrm{e}^{h_w}\varepsilon\right)\left(\left(\frac{s}{\phi}\vec{h}_v + c\right) + \left(\frac{s}{\phi}\vec{d}_v + \frac{c - \frac{s}{\phi}}{\phi^2}\gamma\vec{h}_v - \frac{s}{\phi}\gamma\right)\varepsilon\right) \tag{25}$$

By applying the Taylor series in (24), we can stably evaluate (25) in the neighborhood of $\phi = 0$.

Singularity-Free Dual Quaternion Logarithm We derive the dual quaternion logarithm by expanding the ordinary form (13) with dual arithmetic.

$$\ln\mathcal{S} = \frac{\tilde{\phi}}{\tilde{n}}\left(h_v + d_v\varepsilon\right) + \ln\tilde{m} \tag{26}$$

where $\tilde{\phi}$, $\tilde{n}$, and $\tilde{m}$ are the dual number forms of ϕ, $|h_v|$, and $|h|$, (respectively) from (13). The dual arithmetic expands as follows:

$$\begin{gathered}\tilde{n} = \sqrt{(h_x + d_x\varepsilon)^2 + (h_y + d_y\varepsilon)^2 + (h_y + d_z\varepsilon)^2} = |h_v| + \frac{\vec{h}_v \cdot \vec{d}_v}{|h_v|}\varepsilon = |h_v| + n_d\varepsilon, \\ \tilde{m} = |h| + \frac{h \cdot d}{|h|}\varepsilon = |h| + m_d\varepsilon, \qquad \text{and} \qquad \tilde{\phi} = \tan^{-1}\frac{|h_v| + n_d\varepsilon}{h_w + d_w\varepsilon}\ . \end{gathered} \tag{27}$$

Further expanding $\tilde{\phi}$ via the dual Taylor series for $\tan^{-1}$ (see Table 2):

$$\tilde{\phi} = \operatorname{atan2}\left(|h_v|, h_w\right) + \left(\frac{h_w n_d - |h_v|\, d_w}{|h|^2}\right)\varepsilon = \phi + \left(\frac{h_w n_d - |h_v|\, d_w}{|h|^2}\right)\varepsilon\,. \quad (28)$$

Next, we consider the dual $\frac{\tilde{\phi}}{\tilde{n}}$. For this step we take guidance from the quaternion trigonometry (see Fig. 1) to rewrite factors as trigonometric functions for which we can find well-defined Taylor series. Specifically, we know that:

$$\phi = \operatorname{atan2}\left(h_w, |h_v|\right) \qquad \text{and} \qquad \frac{h_w}{|h|} = \cos\phi \qquad \text{and} \qquad \frac{|h_v|}{|h|} = \sin\phi\,. \quad (29)$$

We expand the dual arithmetic and reorder $\frac{\tilde{\phi}}{\tilde{n}}$:

$$\frac{\tilde{\phi}}{\tilde{n}} = \frac{\phi + \left(\frac{h_w n_d - |h_v| d_w}{|h|^2}\right)\varepsilon}{|h_v| + n_d\varepsilon} = \frac{\phi}{|h_v|} + \left(\frac{h_w n_d}{|h_v|\,|h|^2} - \frac{\phi n_d}{|h_v|^2} - \frac{d_w}{|h|^2}\right)\varepsilon\,. \quad (30)$$

Equation (30) contains a singularity where $|h| = 0$. We evaluate the term $\frac{\phi}{|h_v|}$ as in (14). We rewrite the larger term in the dual coefficient as follows:

$$\frac{h_w n_d}{|h_v|\,|h|^2} - \frac{\phi n_d}{|h_v|^2} = \gamma\left(\frac{h_w}{|h_v|^2\,|h|^2} - \frac{\phi}{|h_v|^3}\right) = \frac{\gamma}{|h|^3}\left(\frac{h_w}{|h|}\frac{|h|^2}{|h_v|^2} - \frac{\phi\,|h|^3}{|h_v|^3}\right)\,. \quad (31)$$

where $\gamma = \vec{h}_v \cdot \vec{d}_v$. Then, we substitute the trigonometric functions and produce the corresponding Taylor series:

$$\frac{\gamma}{|h|^3}\left(\frac{\cos\phi}{\sin^2\phi} - \frac{\phi}{\sin^3\phi}\right) = \frac{\gamma}{|h|^3}\left(-\frac{2}{3} - \frac{1}{5}\phi^2 - \frac{17}{420}\phi^4 + \ldots\right)\,. \quad (32)$$

Now that we have identified Taylor series to handle the singularities, we have the full dual quaternion logarithm:

$$\ln\mathcal{S} = \frac{\phi}{|h_v|}\vec{h}_v + \ln|h| + \left(\frac{\left(\vec{h}_v\cdot\vec{d}_v\right)\alpha - d_w}{|h|^2}\vec{h}_v + \frac{\phi}{|h_v|}\vec{d}_v + \frac{h \cdot d}{|h|^2}\right)\varepsilon$$

$$\text{where} \quad \alpha = \frac{h_w - \frac{\phi}{|h_v|}\,|h|^2}{|h_v|^2} = \frac{\left(-\frac{2}{3} - \frac{\phi^2}{5} - \frac{17\phi^4}{420} + \ldots\right)}{|h|} \quad (33)$$

4.3 Quaternion-Translations as Implicit Dual Quaternions

Just as we may represent transformations with a rotation matrix and translation vector—i.e., the homogeneous transformation matrix—we can also represent transformations with a rotation quaternion and translation vector. The quaternion-translation form offers computational advantages: it consists of only seven elements and chaining requires fewer operations than both the dual quaternion and matrix forms. However, because chaining is no longer a multiplication, as with dual quaternions or matrices, analysis of quaternion-translation kinematics is more complicated, particularly for differential cases involving finding derivatives or in integrating transforms. We address the analytic challenge of the quaternion-translation form by reinterpreting quaternion-translations as *implicit dual quaternions*, or alternately stated, by adopting an in-memory representation of dual quaternions as a quaternion-translation. The implicit dual quaternion combines the analytic convenience of dual quaternions and the computational efficiency the quaternion-translation representation.

The quaternion-translation form stores separately the rotation quaterion $\hbar$ and translation vector $\vec{v}$, eliminating the coupling of rotation and translation in the dual part of the dual quaternion:

$$\underbrace{\hbar + \frac{1}{2}\vec{v} \otimes \hbar\varepsilon}_{\text{explicit dual quaternion}} \quad \overset{\text{rewrite}}{\rightsquigarrow} \quad \underbrace{\begin{pmatrix} \hbar \\ \vec{v} \end{pmatrix}}_{\text{implicit dual quaternion}} \tag{34}$$

To transform a point, we first apply the rotation, then add the translation—the same operations performed by the homogenous transformation matrix:

$$^{a}p = {}^{a}\hbar_{b} \otimes {}^{b}p \otimes ({}^{a}\hbar_{b})^{*} + {}^{a}\vec{v}_{b} \tag{35}$$

Implicit Exponential We derive the exponential for the implicit dual quteration starting with (25), extracting the translation, and finally identifying Taylor series.

First, we simplify (25) to the pure case, i.e., zero scalar part:

$$\begin{gathered} \exp\left(\vec{\omega} + \vec{\nu}\varepsilon\right) = \left(\frac{s}{\phi}\vec{\omega} + c\right) + \left(\frac{s}{\phi}\vec{\nu} + \frac{c - \frac{s}{\phi}}{\phi^2}\gamma\vec{\omega} - \frac{s}{\phi}\gamma\right)\varepsilon \\ \text{where} \quad \gamma = \vec{\omega} \cdot \vec{\nu} \quad \text{and} \quad \phi = \sqrt{\vec{\omega} \cdot \vec{\omega}} \end{gathered} \tag{36}$$

Next, we extract the translation from the dual part.

$$\check{\exp}\left(\vec{\omega} + \vec{\nu}\varepsilon\right) = \begin{pmatrix} \hbar \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \left(\frac{s}{\phi}\vec{\omega} + c\right) \\ 2\left(\frac{s}{\phi}\vec{\nu} + \frac{c - \frac{s}{\phi}}{\phi^2}\gamma\vec{\omega} - \frac{s}{\phi}\gamma\right) \otimes \left(\frac{s}{\phi}\vec{\omega} + c\right)^{*} \end{pmatrix} \tag{37}$$

In (37), we may evaluate the rotation part $\hbar$ as in the ordinary quaternion case. For the translation part $\vec{v}$, we first algebraically simplify:

$$\vec{v} = 2\left(\frac{s}{\phi}\vec{\nu} + \frac{c - \frac{s}{\phi}}{\phi^2}\gamma\vec{\omega} - \frac{s}{\phi}\gamma\right) \otimes \left(\frac{s}{\phi}\vec{\omega} + c\right)^{*} \tag{38}$$

$$= 2\left(-\frac{s^2}{\phi^2}\vec{\nu} \times \vec{\omega} + \frac{cs}{\phi}\vec{\nu} + \frac{c(c - \frac{s}{\phi}) + s^2}{\phi^2}\gamma\vec{\omega}\right) \tag{39}$$

Then, we simplify the trigonometric factors, and we identify the common subexpressions.

$$\vec{\nu} = \frac{2s}{\phi}\left(\left(\frac{s}{\phi}\vec{\omega}\right)\times\vec{v}\right) + c\frac{2s}{\phi}\vec{v} + \left(\frac{2 - c\frac{2s}{\phi}}{\phi^2}\right)\gamma\vec{\omega} \tag{40}$$

Using the Taylor series from Table 1 and for the new factor, we obtain:

$$\breve{\exp}(\vec{\omega} + \vec{\nu}\varepsilon) = \begin{pmatrix} h \\ \vec{v} \end{pmatrix} = \begin{bmatrix} \left(\frac{s}{\phi}\vec{\omega} + c\right) \\ \frac{2s}{\phi}\left(\vec{h}_v \times \vec{\nu}\right) + c\frac{2s}{\phi}\vec{\nu} + \left(\frac{2-c\frac{2s}{\phi}}{\phi^2}\right)(\vec{\omega}\cdot\vec{\nu})\,\vec{\omega} \end{bmatrix}$$

$$\frac{s}{\phi} = 1 - \frac{\phi^2}{6} + \frac{\phi^4}{120} + \ldots \quad \text{and} \quad \frac{2 - c\frac{2s}{\phi}}{\phi^2} = \frac{4}{3} - \frac{4\phi^2}{15} + \frac{8\phi^4}{315} + \ldots \tag{41}$$

Implicit Logarithm We derive the implicit logarithm starting with (33), substituting the translation vector, and finally identifying suitable Taylor series.

We begin with the dual quaternion logarithm (33):

$$\breve{\ln}\begin{pmatrix} h \\ \vec{v} \end{pmatrix} = \vec{\omega} + \vec{\nu}\varepsilon = \ln\left(h + \frac{1}{2}\vec{v}\otimes h\varepsilon\right) \tag{42}$$

The real part $\vec{\omega}$ of the implicit logarithm is identical to the dual quaternion case (33). We assume a unit quaternion $|h| = 1$, so the scalar part of the logarithm is zero.

$$\left(\breve{\ln}\begin{pmatrix} h \\ \vec{v} \end{pmatrix}\right)_{\text{real}} = \vec{\omega} = \frac{\phi}{|h_v|}\vec{h}_v = \frac{\phi}{\sin\phi}\vec{h}_v \tag{43}$$

For the dual part $\vec{\nu}$, we expand (33), simplifying for the unit case $|h| = 1$:

$$\left(\breve{\ln}\begin{pmatrix} h \\ \vec{v} \end{pmatrix}\right)_{\text{dual}} = \vec{\nu} = (\gamma\alpha - d_w)\,\vec{h}_v + \frac{\phi}{|h_v|}\vec{d}_v + \gamma + h_w d_w$$

$$\text{where} \quad \vec{d}_v = \frac{1}{2}\vec{v}\times\vec{h}_v + \frac{1}{2}h_w\vec{v} \quad \text{and} \quad d_w = -\frac{1}{2}\vec{v}\cdot\vec{h}_v \tag{44}$$

Substituting for dual part d in terms of translation $\vec{v}$, we simplify to:

$$\vec{\nu} = -\frac{1}{2}\vec{v}\cdot\vec{h}_v\left(\frac{\frac{h_w\phi}{s} - 1}{s^2}\right)\vec{h}_v + \frac{h_w\phi}{2s}\vec{v} + \frac{\phi}{2s}\left(\vec{v}\times\vec{h}_v\right) \tag{45}$$

Noting that $h_w = \cos\phi$ and $\vec{\omega} = \frac{\phi}{\sin\phi}\vec{h}_v$, we further simplify to:

$$\vec{\nu} = \left(\frac{\vec{v}}{2}\right)\cdot\vec{\omega}\left(\frac{1 - c\frac{\phi}{s}}{\phi^2}\right)\vec{\omega} + c\frac{\phi}{s}\left(\frac{\vec{v}}{2}\right) + \left(\left(\frac{\vec{v}}{2}\right)\times\omega\right) \tag{46}$$

Finally, we identify the Taylor series to obtain the implicit logarithm as follows:

$$\breve{\ln}\begin{pmatrix} \vec{h} \\ \vec{v} \end{pmatrix} = \vec{\omega} + \vec{\nu}\varepsilon$$
$$= \frac{\phi}{\sin\phi}\vec{h}_v + \left(\frac{\vec{v}}{2}\cdot\vec{\omega}\left(\frac{1-c\frac{\phi}{s}}{\phi^2}\right)\vec{\omega} + \left(\frac{c\phi}{s}\right)\frac{\vec{v}}{2} + \frac{\vec{v}}{2}\times\omega\right)\varepsilon$$
$$\frac{c\phi}{s} = 1 - \frac{\phi^2}{3} - \frac{\phi^4}{45} - \ldots \quad \text{and} \quad \frac{1-\frac{c\phi}{s}}{\phi^2} = \frac{1}{3} + \frac{\phi^2}{45} + \frac{2\phi^4}{945} + \ldots \tag{47}$$

5 Application to Kinematics

Both matrix and quaterion representations may be used to compute the forward kinematics of robot manipulators. We compare the different representations and show that the quaternion-translation offers the best forward kinematics performance. Mathematically, quaternion-translations require the fewest arithmetic operations, and in our empirical evaluation, quaternion-translations require the shortest execution time.

Table 3 and Table 4 compare operations for quaternion and matrix forms.

Table 5 summarizes the construction of transformations for single degree-of-freedom joints using matrix and quaternion forms. We use the known axis of joints to simplify construction over the general-case exponential. The result shows that quaternion-translations require the fewest arithmetic operations.

Fig. 2 presents an empirical comparison of forward kinematics performance in terms of speedup over the baseline matrix representation. The quaternion-translation shows the best empirical performance, consistent with the operation counts in Table 3, Table 4, and Table 5. Additionally, the explicit dual quaternion also offers slightly better performance than matrices in our tests. Even though matrices require fewer arithmetic operations to construct and chain, several other advantages of the dual quaternions lead to the improved performance. Dual quaternions are more compact than matrices, which reduces necessary data shuffling, and quaternions require fewer operations for the exponential and rotation chaining, which are heavily used in robots with many revolute frames.

6 Discussion

We often have the choice of a matrix or quaternion form for any particular application; both will produce a mathematically-equivalent result, but the computational efficiency will differ. For example, interpolation is commonly regarded as a key application area for quaternions; however, we can achieve the same result—at greater computational cost—using rotation matrices. Spherical linear interpolation (SLERP) [20] interpolates from an initial to final orientation

			Chain		Rot./Tf.		Normalize		
	Representation	*Storage*	*Mul.*	*Add*	*Mul.*	*Add*	*Mul.*	*Add*	*Other*
Rot.	*Rotation Matrix*	9	27	18	9	6	27	15	sqrt(3)
	Quaternion	4	16	12	15	15	8	3	sqrt
Tf.	*Transformation Matrix*	12	36	27	9	9	27	15	sqrt(3)
	Dual Quaternion	8	48	40	28	28	12	3	sqrt
	Quaternion-Translation	7	31	30	15	18	8	3	sqrt

Table 3. Requirements for storage, chaining, and point transformation. Quaternion-based representations are more compact than matrices. Ordinary quaternions and quaternion-translations are most efficient for chaining rotations and transformations, respectively. Matrices are most efficient for rotating and transforming points.

		Exponential			Logarithm		
	Representation	*Mul.*	*Add*	*Other*	*Mul.*	*Add*	*Other*
Rot.	*Rot. Matrix*	17	15	sqrt, sincos	5	7	sqrt, atan2
	Quaternion	9	2	sqrt, sincos, exp	8	3	sqrt(2), atan2, ln
	Unit Q.	7	2	sqrt, sincos	7	2	sqrt, atan2
Tf.	*Tf. Matrix*	39	34	sqrt, sincos	31	32	sqrt, atan2
	Dual Quat.	31	12	sqrt, sincos, exp	22	11	sqrt(2), atan2, ln
	Unit Dual Q.	19	8	sqrt, sincos	18	9	sqrt, atan2
	Quat.-Trans.	28	15	sqrt, sincos	28	16	sqrt, atan2

Table 4. Exponential and Logarithm Operation Counts. Ordinary and dual quaternions are more efficient than their matrix equivalents. The quaternion-translation costs are between the matrix and dual-quaternion.

		Form	*Mul.*	*Add*	*Other*
Revolute	*Tf. Matrix*	$\begin{bmatrix} e^{[\theta\hat{u}]} & \mathbf{v} \\ 0 & 1 \end{bmatrix}$	12	13	sincos
	Dual Quat.	$e^{\frac{\theta}{2}\hat{u}} + \frac{\vec{v}}{2} \otimes e^{\frac{\theta}{2}\hat{u}} \boldsymbol{\varepsilon}$	19	12	sincos
	Quat.-Trans.	$\begin{pmatrix} e^{\frac{\theta}{2}\hat{u}} \\ \vec{v} \end{pmatrix}$	3	0	sincos
Prismatic	*Tf. Matrix*	$\begin{bmatrix} \mathbf{R} & \ell\hat{\mathbf{u}} \\ 0 & 1 \end{bmatrix}$	3	0	-
	Dual Quat.	$\hbar + \ell\left(\frac{\hat{u}}{2} \otimes \hbar\right)\boldsymbol{\varepsilon}$	4	0	-
	Quat.-Trans.	$\begin{pmatrix} \hbar \\ \ell\hat{u} \end{pmatrix}$	3	0	-
Helical	*Tf. Matrix*	$\begin{bmatrix} e^{[\theta\hat{u}]} & (k\hat{\mathbf{u}})\,\theta \\ 0 & 1 \end{bmatrix}$	15	13	sincos
	Dual Quat.	$e^{\frac{\theta}{2}\hat{u}} + \theta\frac{k\hat{u}}{2} \otimes e^{\frac{\theta}{2}\hat{u}} \boldsymbol{\varepsilon}$	23	14	sincos
	Quat.-Trans.	$\begin{pmatrix} e^{\frac{\theta}{2}\hat{u}} \\ (k\hat{u})\theta \end{pmatrix}$	6	0	sincos

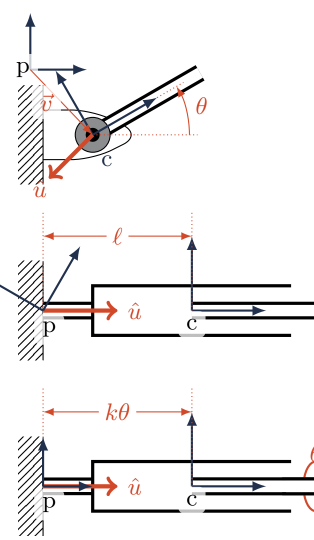
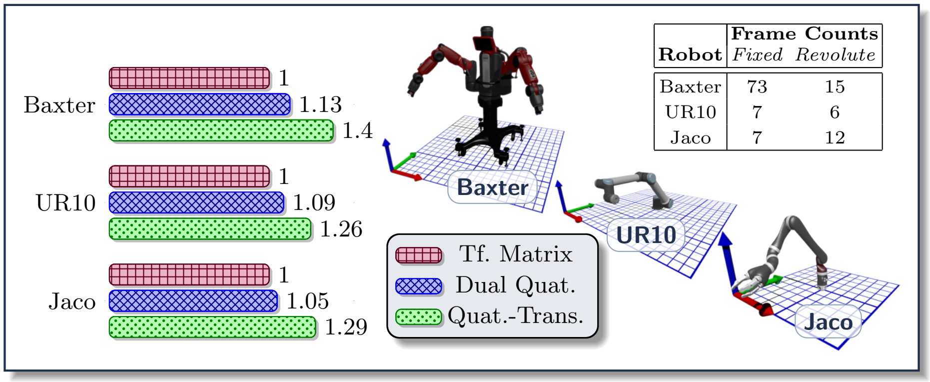


Table 5. Single degree-of-freedom joint transforms and operation counts. The quaternion-translation representation is most efficient to construct.



Robot	Frame Counts *Fixed*	*Revolute*
Baxter	73	15
UR10	7	6
Jaco	7	12

Fig. 2. Forward kinematics speedup (higher is better), demonstrating a 25%-40% performance improvement using quaternion forms. We compare the execution time to compute the forward kinematics for the Rethink Baxter, Universal Robots UR10, and Kinova Jaco manipulators using transformation matrices, dual quaternions, and quaternion-translations. The results are shown as speedup ($\frac{t_{\mathrm{baseline}}}{t_{\mathrm{new}}}$) over the transformation matrix case. The tests were conducted on an Intel® Xeon E3-1275 v6 using the kinematics implementations in the Amino (`http://amino.dyalab.org`) library.

with constant rotational axis and linearly-varying angle. The algebraic form of SLERP [3] has a direct matrix equivalent:

$$\hbar(\tau) = \hbar(0) \otimes \exp\left(\tau \ln\left((\hbar(0))^* \otimes \hbar(1)\right)\right) \tag{48}$$

$$\mathbf{R}(\tau) = \mathbf{R}(0)\ \exp\left(\tau \ln\left((\mathbf{R}(0))^{-1}\ \mathbf{R}(1)\right)\right) \tag{49}$$

where $\hbar(0), \mathbf{R}(0)$ is the initial orientation and $\hbar(1), \mathbf{R}(1)$ is the final orientation. Both (48) and (49) equivalently interpolate orientation. However, the quaternion form (48) is more efficient to compute than the matrix form, and the more commonly used geometric simplification of (48) is even more efficient [20].

Similarly, ordinary and dual quaternions provide computational advantages for the blending or averaging of rotations and transformations [11,15], which was described by Wahba [24] as optimal rotation based on a set of weighted observations.

Generally, the matrix and quaternion representations of rotation and Euclidean transformation share group structure. Just as the rotation and transformation matrices form Lie groups with associated Lie algebras based on the exponential, so too do the ordinary and dual quaternions form Lie groups and associated algebras. We can map every quaternion representation to matrix equivalent. Specifically, there is a surjective homomorphism (double-cover) from the ordinary unit quaternions to the special orthogonal group $\mathcal{SO}(3)$ of rotation matrices. Similarly, we have a surjective homomorphism from the dual unit quaternions to the special Euclidean group $\mathcal{SE}(3)$ of homogeneous transformation matrices.

The results we have presented continue the broader developments of methods based on ordinary and dual quaternions which offer computational advantages over their matrix counterparts. The quaternion methods we have presented achieve mathematically-equivalent results, but are more compact and efficient, than the matrices.

7 Conclusion

We have presented new derivations of the dual quaternion exponential and logarithm which handle the small-angle singularity and enable the use of dual quaternions within the product of exponentials formulation of robot kinematics. By extending our singularity-robust exponential and logarithm to the implicit representation of dual quaternions as an ordinary quaternion and translation vector, we demonstrate a 25%-40% performance improvement in kinematics computation over the conventional homogeneous transformation matrices. Our implementation is available as open source code[2]. These results show that dual quaternion representations provide the same capabilities as transformation matrices and offer computational advantages which may be especially useful for resource-constrained systems.

Even though matrices are a widely-used representation for Euclidean transformations, the quaternion forms are both more compact and—for most cases—require fewer arithmetic operations. In the one case where matrices have an efficiency advantage—transforming large numbers of points—it may still be more efficient to chain transformations via quaternions and then convert the final transform to a matrix to apply to the point set. We hope these derivations of singularity-free exponentials and logarithms for the quaternion forms of transformations will enable widespread use of these more efficient representations.

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[2] Software available at `http://amino.dyalab.org`

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Errata

- Equation 5: corrected third partial term from $|\omega|^2$ to $|\omega|^4$.
- Table 2: added missing closed parenthesis to first line.
- Equation 47: corrected third coefficient from $\frac{1}{945}$ to $\frac{2}{945}$.
- Equation 18: corrected real part from ${}^{a}\hbar_{b} \otimes {}^{a}\hbar_{b}$ to ${}^{a}\hbar_{b} \otimes {}^{b}\hbar_{c}$.
- Equation 33: corrected third coefficient from $\frac{1}{420}$ to $\frac{17}{420}$.